Good characterization for path packing in a subclass of Karzanov networks

Natalia Vanetik[1]

Abstract

The path packing problem is stated finding the maximum number of edge-disjoint paths between predefined pairs of nodes in an undirected multigraph. Such a multigraph together with predefined node pairs is often called a network. While in general the path packing problem is NP-hard, there exists a class of networks for which the hope of better solution for the path packing problem exists. In this paper we prove a combinatorial max-min theorem (also called a good characterization) for a wide class of such networks, thus showing that the path packing problem for this class of networks is in co-NP.

Keywords: integer path packing

1. Introduction

A *network* consists of a multigraph $G = (N, E)$, sometimes called *supply graph*, a set of *terminals* $T \subseteq N$ and a *demand graph* (T, S) with $S \subseteq T \times T$. Let $\mathcal{K}$ denote the collection of inclusion-maximal anticliques of the graph (T, S) (such a collection is called a *clutter*). $\mathcal{K}$ divides the pairs of terminals into three classes: pairs not covered by a member of $\mathcal{K}$ (this is precisely S), pairs covered by a single member of $\mathcal{K}$ (we denote this class W) and pairs covered by more than one member of $\mathcal{K}$ (called *equivalent pairs*). Therefore, a network $G, (T, S)$ can be also defined by specifying G, T and $\mathcal{K}$ and denoted $(G, T, \mathcal{K})$. In this paper we study a subclass of *Eulerian networks* where all nodes in $N \setminus T$, called *inner nodes*, have even degrees.

We refer to the paths with end-pair in W as *W-paths*, and the paths with the end-pair in S as *S-paths*. Collection of edge-disjoint paths with ends in T is called a *multiflow*. Various multiflow optimization problems are studied in the literature. A major multiflow optimization problem, called the *(integer) path packing problem*, is to

$$\text{find the maximum number of edge-disjoint } S\text{-paths in } (G, T, \mathcal{K}). \tag{1.1}$$

In this paper we refer to this problem as the *strong problem*. For a multiflow f, we denote by $f[W]$ and $f[S]$ the number of S-paths and W-paths in f. An optimization problem called the *weak problem* is to

$$\text{find the maximum of } f[S] + \tfrac{1}{2} f[W] \text{ over multiflows } f \text{ in } (G, T, \mathcal{K}). \tag{1.2}$$

When paths of a multiflow are allowed to have a fractional weight between 0 and 1 and the total weight of all paths traversing an edge does not exceed 1, we speak of a *fractional multiflow* and

[1]Department of Computer Science, Ben-Gurion University, Israel, *orlovn@cs.bgu.ac.il*

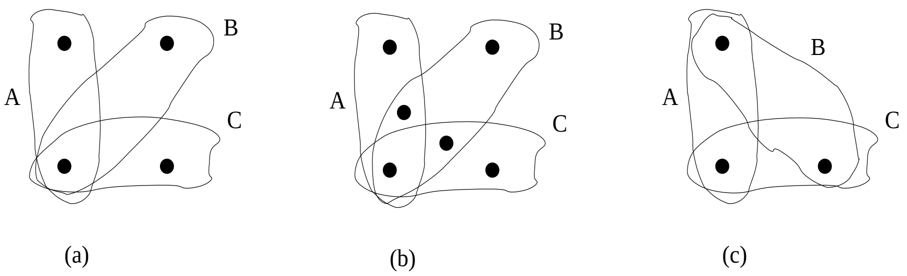


Figure 1: Clutter members intersections in a K-network and non-K-networks.

fractional strong and weak problems. The maxima of fractional and integer strong problems in a network $(G, T, \mathcal{K})$ are further denoted $\eta^{\text{FR}}(G, T, \mathcal{K})$ and $\eta(G, T, \mathcal{K})$, and the maxima of fractional and integer strong problems in $(G, T, \mathcal{K})$ are denoted as $\theta^{\text{FR}}(G, T, \mathcal{K})$ and $\theta(G, T, \mathcal{K})$. When speaking of integer strong or weak problem, we will omit the word "integer" for simplicity. A network $(G, T, \mathcal{K})$ is called *integral* if $\theta^{\text{FR}}(G, T, \mathcal{K}) = \theta(G, T, \mathcal{K})$. Fractional strong problem can be solved by linear programming in polynomial time, strong problem is NP-hard in general.

A. Karzanov has discovered (see [K 1989]) that tractability of the strong problem is determined by the structure of $\mathcal{K}$. He defined a subset of networks for which any two pairwise intersecting anticliques $A, B, C \in \mathcal{K}$ satisfy

$$A \cap B = A \cap C = B \cap C. \tag{1.3}$$

An Eulerian network that satisfies (1.3) is called a *K-network*. Figure 1(a) shows an example of K-network, and Figure 1(b)-(c) shows two examples of networks where condition 1.3 does not hold. It has been shown in [BI 2003] that the strong problem is NP-hard if (1.3) is not satisfied, even if the network is Eulerian. The weak problem has convenient metric properties and can be used as a proof tool because of the connection it shares with the strong problem in K-networks, as stated in the following theorem.

Theorem 1.1. [V 2007] *In a K-network, strong problem and weak problem have a common solution.*

Throughout the paper, we will call a multiflow solving both strong and weak problem in a network *a common solution*.

This paper has following structure. Section 2 describes the problem addressed in this paper and gives historical background. Section 3 describes the main combinatorial structure, and Section 4 contains the extended proof of the main result. This result is used in Section 5 to show that the addressed problem is in co-NP. The Appendix contains definitions and properties of path operations used in the proof and the theory of dual solutions of the weak problem in K-networks. Table 1 summarizes notation used in this paper.

2. Problem statement

When one wishes to solve a multiflow optimization problem such as the strong problem, the ultimate goal is to construct a polynomial-time algorithm that finds a solution. In absence of such algorithm, the next best thing is combinatorial *max-min theorem* of the form:

$$\max_f f[S] = \min_C \varphi(C),$$

Notation	Definition
$(G,T,\mathcal{K})$	a network $(G,(T,S))$ and anticlique clutter $\mathcal{K}$ of (T,S)
S-path (W-path)	a path whose end-pair is in S (in W)
$f[S]$ ($f[W]$)	the number of S-paths (W-paths) in multiflow f
A^c, $A \subseteq T$	$T \setminus A$
$\overline{A}$, $A \subseteq N$	$N \setminus A$
$d(X)$, $X \subset N$	the number of $(X,\overline{X})$-edges in G
$\lambda(A)$, $A \subseteq T$	size of a minimal (A,A^c)-cut
$\beta(A)$, $A \subseteq T$	$\frac{1}{2}(\sum_{t\in A}\lambda(t)-\lambda(A))$
(A,B)-path (A-path), $A,B \subseteq T$	path with ends in A and B (in A)
$f[A,B]$ ($f[A]$	the number of (A,B)-paths (A-paths) in multiflow f
$\Theta(f)$	$f[S]+\frac{1}{2}f[W]$
xPy, $x,y \in N$	(x,y)-segment of path P
$\|f\|$	the size of multiflow f, i.e. the number of its paths
a maximum multiflow	a multiflow of maximum size (i.e. size is $\frac{1}{2}\sum_{t\in T}\lambda(t)$)
$\mathcal{K}$ is simple	$\|A\cap B\| \leq 1$ for all $A,B \in \mathcal{K}$
$\mathcal{K}$ is flat	$\|A\| = 2$ for all $A \in \mathcal{K}$
T-path P with ends s,t is compound	$\exists r \in T$ s.t. $r \in P$, $r \neq s,t$

Table 1: Notation.

where the maximum is taken over multiflows in a network, the minimum is taken over combinatorial structures C in a network, and φ is a function computable on C in time polynomial in the size of the network. Such a theorem is sometimes called a *good characterization*, notion introduced by J. Edmonds (see [E 1965]). In other words, a theorem is a good characterization for a problem if it gives necessary and sufficient conditions for the existence of a certain structure in terms of the absence of another structure. For example, the existence of an integer flow of size n between the two terminals t and s (in other words, the existence of n edge-disjoint paths between those terminals) is guaranteed by the absence of a (t,s)-mincut of capacity less than n.

Over the years, combinatorial max-min theorems for the path packing problem were proved for certain special cases of networks, such as:

1. S is a single edge (Menger's theorems [Me 1927]) for general networks,
2. S consists of two edges ([Hu 1963]), for Eulerian networks,
3. S is a complete graph, ([Ch 1977, Lo 1976]), for Eulerian networks,
4. Every $t \in T$ is covered by at most two members of $\mathcal{K}$, (in [IKL 2000]), for Eulerian networks,
5. $W \cong K_{2,r}$, with an arbitrary r, ([L 2004]), for Eulerian networks,
6. S is a complete graph (Mader's edge-disjoint path-packing theorem [Ma 1978]), for general networks.

Let us call an anticlique clutter $\mathcal{K}$ of a network $(G,T,\mathcal{K})$ *flat* if $\mathcal{K}$ satisfies condition (1.3) and each member of $\mathcal{K}$ contains precisely two terminals. Thus, $\mathcal{K}$ is flat if and only if it coincides with the edge set of a triangle-free graph. A network $(G,T,\mathcal{K})$ is called *flat* if its anticlique clutter $\mathcal{K}$ is flat. Note that every flat network is a K-network, but the converse is not always true.

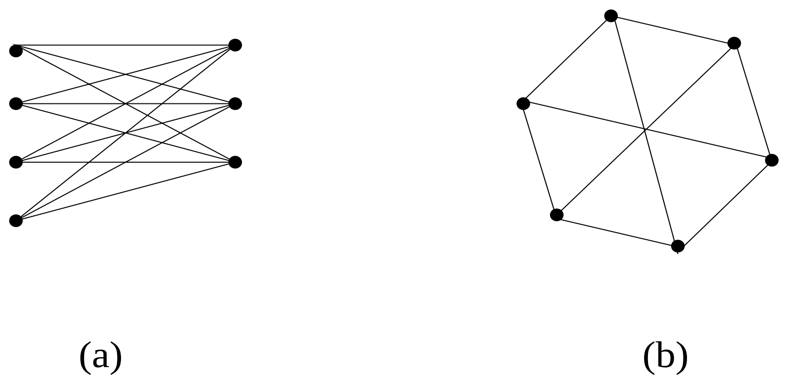


Figure 2: Examples of W in flat networks.

In this paper, we focus on flat networks. This class contains classes 1-3 and 4-5 for Eulerian networks and has a non-trivial intersection with class 4 of networks. We define a combinatorial structure consisting of *terminal expansions* (node subsets containing precisely one terminal each) and a clutter on T, and show the existence of a function on this structure that gives a good characterization for the fractional strong problem. In integral flat networks, our theorem is a good characterization for the strong problem, because a least-size integer solution to the weak problem in the expanded network also solves the strong problem, by Theorem 1.1. Figure 2 shows the structure of W as a graph on T in two flat networks.

3. Dual combinatorial structure

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two clutters on T. We say that $\mathcal{K}_2$ *extends* $\mathcal{K}_1$ and denote the fact by $\mathcal{K}_1 \preceq \mathcal{K}_2$ if every pair of terminals that is strong in $\mathcal{K}_1$ is also strong in $\mathcal{K}_2$. Relation $\preceq$ is transitive. We use here the definition of *expansion of* T *in a graph* G as a collection $\mathcal{X} = \{X_t \subseteq N \mid t \in T\}$ of pairwise disjoint subsets of N covering T, with every subset X_t containing exactly one terminal. We define a partial order on expansions in T by saying that for two expansions $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \preceq \mathcal{Y}$ if for every $X \in \mathcal{X}$ there exists $Y \in \mathcal{Y}$ so that $X \subset Y$. Our combinatorial structure for network $(G, T, \mathcal{K})$ will include both an expansion of T and an extension of $\mathcal{K}$. A network obtained from $(G, T, \mathcal{K})$ by replacing each terminal in T with its expansion member in $\mathcal{X}$ is denoted $(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}})$. Examples of clutter extension and terminal expansions are given in Figure 3.

Intuitively, we are looking for a way to expand terminals and extend $\mathcal{K}$ until we obtain a network where a maximum solution to the weak problem locks all the members of the expansion and the induced clutter. Among all such networks, we are looking for one where a least-size solution to the weak problem has as many S-paths as the strong problem solution in $(G, T, \mathcal{K})$.

As a first step of showing that the above dual combinatorial structure gives us a good characterization for the strong problem, we prove the following.

Theorem 3.1. *For every flat network* $(G, T, \mathcal{K})$*, and every flat clutter* $\mathcal{R} \succeq \mathcal{K}$ *on* T*, and every expansion* $\mathcal{X}$ *of* T *in* G *we have* $\eta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{R}_{\mathcal{X}}) \geq \eta(G, T, \mathcal{K})$*.*

Proof. Let us consider an arbitrary flat clutter $\mathcal{R} \succeq \mathcal{K}$ on T and an arbitrary expansion $\mathcal{X}$ of T. Expanding terminals does not decrease $\eta(G, T, \mathcal{K})$ because an expansion member contains just one terminal. Thus, for every multiflow h in $(G, T, \mathcal{K})$, an S-path of h remains an S-path and thus $\eta(G, T, \mathcal{K})$ does not decrease. Likewise, an S-path of h remains an S-path in $(G, T, \mathcal{R})$, thus $\eta(G, T, \mathcal{R}) \geq \eta(G, T, \mathcal{K})$ and $\eta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{R}_{\mathcal{X}}) \geq \eta(G, T, \mathcal{K})$. □

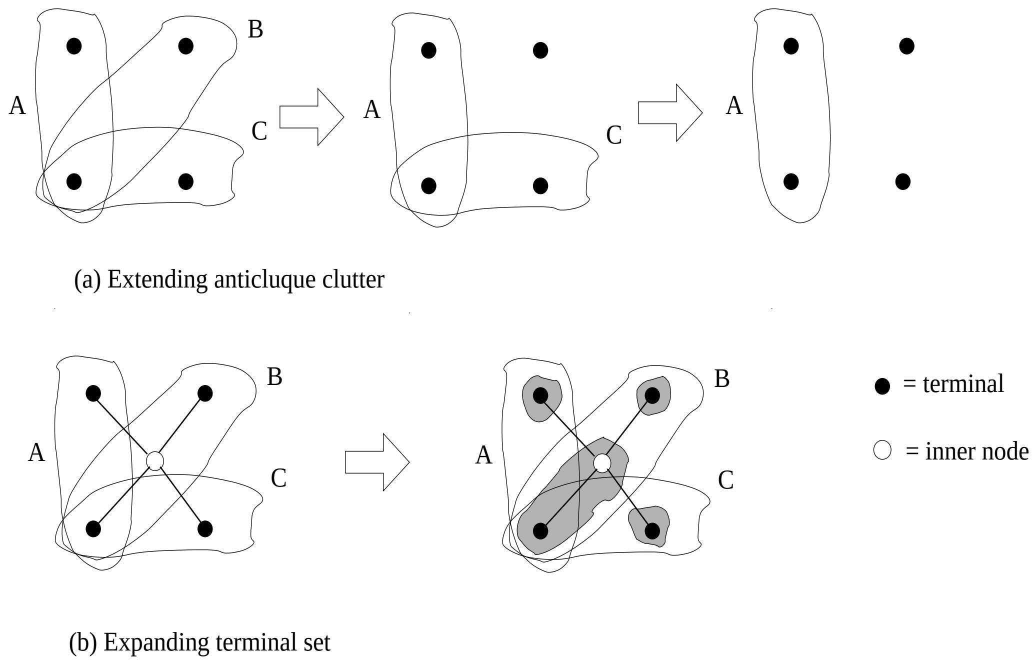


Figure 3: Examples of extension and expansion.

The next step is to prove that for every flat $(G, T, \mathcal{K})$, there exists a flat clutter $\mathcal{R} \succeq \mathcal{K}$ and an expansion of T in $(G, T, \mathcal{R})$, which is also a dual solution of the weak problem in $(G, T, \mathcal{R})$, such that the maxima of the fractional strong problem, denoted η^{FR}, in the original and expanded network are the same. Let us define the *fractionality* of a network as the value of minimal path weight denominator over all weak problem solutions in that network (if a network is integral, its fractionality equals 1).

Theorem 3.2. *For every flat network $(G, T, \mathcal{K})$, there exists a flat clutter $\mathcal{R} \succeq \mathcal{K}$ on T and a dual solution $\mathcal{X}$ in $(G, T, \mathcal{R})$ such that $\eta^{\mathrm{FR}}(G_{\mathcal{X}}, \mathcal{X}, \mathcal{R}_{\mathcal{X}}) = \eta^{\mathrm{FR}}(G, T, \mathcal{K})$ and $(G, T, \mathcal{R})$ has the same fractionality as $(G, T, \mathcal{K})$.*

We will later use Theorems 3.1 and 3.2 to show that $(\mathcal{R}, \mathcal{X})$ as a dual combinatorial structure gives a good characterization for the fractional strong problem in flat networks and for the strong problem in integral flat networks.

Let us demonstrate by example that $\mathcal{R}$ is a necessary parameter of a dual combinatorial structure, or, in other words, that the simple method of expansions suitable for the weak problem is insufficient for the strong problem. Consider network $(G, T, \mathcal{K})$ of Figure 4, where every common solution unlocks A. An inner node x thus must be contained in any dual solution $\mathcal{X}$ of $(G, T, \mathcal{K})$. Adding x to any X_s, $s \neq t$, increases $\eta(G, T, \mathcal{K})$. Adding x to X_t creates a network $(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}})$ with $\eta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}}) = 3$, as (A_i, B_i)-paths for $i = 1, 2, 3$ are strong paths. Since $\eta(G, T, \mathcal{K}) = 2 < \eta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}})$, a simple method of expansions does not work here. This problem can be solved by extending $\mathcal{K}$ into $\mathcal{R} := \mathcal{K} \setminus \{A_1\}$.

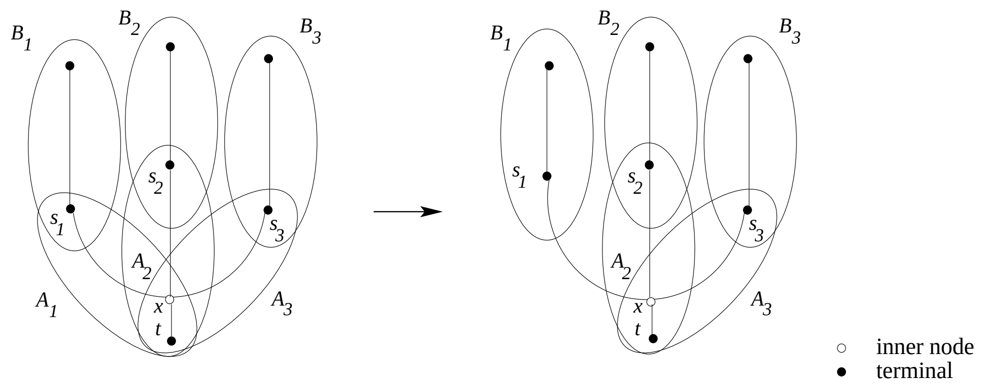


Figure 4: Clutter extension.

4. Proof of Theorem 3.2

4.1. A minimal counterexample network

To prove Theorem 3.2, we select a counterexample integral flat network $(G, T, \mathcal{K})$ minimal w.r.t.

$$N \setminus T,\ E \textit{ and the clutter extension.}$$

We can assume that the weak problem admits an integer solution in $(G, T, \mathcal{K})$ as the necessary network can always be achieved by multiplying the edges of E. We are, therefore, looking for integral network $(G, T, \mathcal{R})$. A flat clutter $\mathcal{K}$ can be extended and remain flat by removing a clutter member. If a flat clutter $\mathcal{K}$ cannot be extended, $\mathcal{K} = \emptyset$ and thus $S = T \times T$. In this case T and $\mathcal{K}$ satisfy Theorem 3.2.

Let

$$\mathcal{X} \textit{ be the minimal dual solution in } (G, T, \mathcal{K}),$$

which exists by Claim 7.8. Since $(G, T, \mathcal{K})$ is integral, we have $\theta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}}) = \theta(G, T, \mathcal{K})$.

Also, let

$$h \textit{ be a common solution in } (G, T, \mathcal{K}) \textit{ and } g_{\mathcal{X}} \textit{ be a common solution in } (G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}}).$$

Claims 7.6 and 7.7 imply that

$$\hat{h}_{\mathcal{X}} \textit{ locks } \mathcal{X} \textit{ and } \mathcal{K}_{\mathcal{X}}, \textit{ and } \Theta(h_{\mathcal{X}}) = \Theta(\hat{h}_{\mathcal{X}}) = \theta(G, T, \mathcal{K}) = \Theta(g_{\mathcal{X}}). \tag{4.4}$$

Then by minimality of $(G, T, \mathcal{K})$ we have

$$\eta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}}) > \eta(G, T, \mathcal{K}), \tag{4.5}$$

because otherwise $\mathcal{K}$ and $\mathcal{X}$ satisfy conditions of Theorem 3.2.

4.2. Properties of clutter extension

For convenience, we assume that all inner nodes have degree 4 throughout the paper (otherwise, the network can be easily modified to turn have all inner nodes of degree 4 while preserving all flows both ways).

Claim 4.1. *Let $(G, T, \mathcal{K})$ be a flat network, $\mathcal{R} \succ \mathcal{K}$ be a flat clutter and h be a maximum weak problem solution in $(G, T, \mathcal{K})$. Then every augmenting sequence of h in $(G, T, \mathcal{R})$ is an augmenting sequence of h in $(G, T, \mathcal{K})$.*

Proof. Let us assume that h contains an augmenting sequence $P_0, x_0, P_1, ..., x_n, P_n$ for $B \in \mathcal{R}$ in $(G, T, \mathcal{R})$. Then P_0 is a B-path of h in $(G, T, \mathcal{R})$ and P_1 is either a (B, B^c)-path or a B^c-path, depending on n. If P_1 is a (B, B^c)-path in $(G, T, \mathcal{R})$, it could have been a B-path or a (B, B^c)-path of h in $(G, T, \mathcal{K})$. In either case, the rest of the augmenting sequence exists in $(G, T, \mathcal{K})$. If P_1 is a B^c-path in $(G, T, \mathcal{R})$, then P_1 is a B-path or a (B, B^c)-path in $(G, T, \mathcal{K})$. In both cases, since $\mathcal{K}$ is flat and $\mathcal{R} \succ \mathcal{K}$, $B \notin \mathcal{R}$ - a contradiction. $\square$

Corollary 4.2. *Let $(G, T, \mathcal{K})$ be a flat network, $\mathcal{R} \succ \mathcal{K}$ be a flat clutter and h be a maximum weak problem solution in $(G, T, \mathcal{K})$ locking $\mathcal{K}$. Then h locks $\mathcal{R}$.* $\square$

Corollary 4.3. *Let $(G, T, \mathcal{K})$ and $(G, T, \mathcal{R})$ be integral flat networks with $\mathcal{R} \succ \mathcal{K}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be minimal dual solutions in $(G, T, \mathcal{K})$ and $(G, T, \mathcal{R})$ respectively. Then $\mathcal{Y} \preceq \mathcal{X}$.* $\square$

4.3. Proof of Theorem 3.2

Here, we complete the proof of Theorem 3.2. First, we show that no inner node lies outside the minimal dual solution of $(G, T, \mathcal{K})$.

Claim 4.4. $N \setminus \mathcal{X} = \emptyset$.

Proof. Let us first assume an inner node $x \in X_t \in \mathcal{X}$ lies on an A-path of $h_{\mathcal{X}}$. Since both $\hat{h}_{\mathcal{X}}$ and $\hat{g}_{\mathcal{X}}$ lock $\mathcal{K}_{\mathcal{X}}$, x does not lie on an A^c-path in either multiflow. In a flat network it implies that x lies on two paths with a common end (see Figure 5). Therefore, x admits two splits

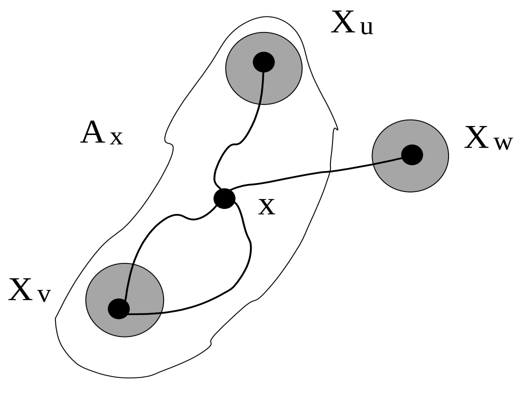


Figure 5: Inner node $x \notin \mathcal{X}$ on an A-path of $\hat{h}_{\mathcal{X}}$ or $\hat{g}_{\mathcal{X}}$.

preserving $\Theta(h)$ and $h[S]$, and two splits preserving $\Theta(g_{\mathcal{X}})$ and $g_{\mathcal{X}}[S]$. Since x can be split in three different ways only, there exists a split of x preserving $\Theta(h)$, $h[S]$ and $g_{\mathcal{X}}[S]$. Let us denote the network obtained by this split by $(G', T, \mathcal{K})$. Then by minimality of $(G, T, \mathcal{K})$ there exists a flat clutter $\mathcal{R} \succeq \mathcal{K}$ and minimal dual solution $\mathcal{Y}$ in $(G', T, \mathcal{R})$ for which Theorem 3.2 holds. From Corollary 4.3 we have $\mathcal{Y} \prec \mathcal{X}$.

Let us show that $\hat{h}_{\mathcal{Y}}$ solves the weak problem in network $(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$ obtained by restoring x. If not, $\hat{h}_{\mathcal{Y}}$ unlocks $B \in \mathcal{R}_{\mathcal{Y}}$ and therefore contains an augmenting sequence $P_0, x_0, P_1, ..., x_n, P_n$

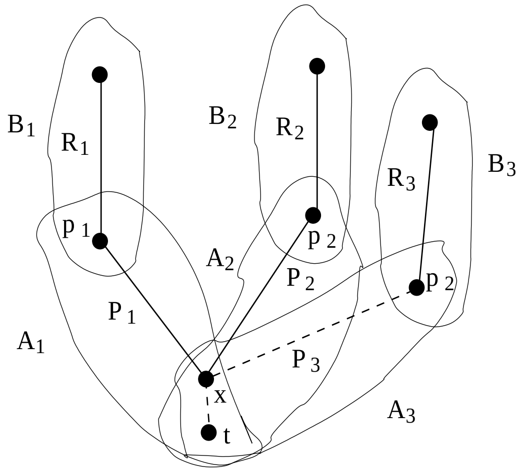


Figure 6: Paths P_i and R_i for $i = 1, 2, 3$.

for B. Since $\hat{h}_{\mathcal{X}}$ locks each $A \in \mathcal{K}_{\mathcal{X}}$ in $(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}})$ and $x \notin \mathcal{X}$, $x = x_i$ for some $0 \leq i \leq n$. By Claim 4.1 this augmenting sequence exists in $(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{K}_{\mathcal{Y}})$ as well, and by Claim 7.7 $x \in \mathcal{X}$ - a contradiction. Therefore, $\hat{h}_{\mathcal{Y}}$ locks each member of $\mathcal{R}_{\mathcal{Y}}$ in $(G'_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$ and $(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$ and thus the value of $\beta(A)$ is the same for all $A \in \mathcal{R}_{\mathcal{Y}}$ in both networks. This also implies that $(G, T, \mathcal{R})$ is integral.

Let $f_{\mathcal{Y}}$ be a common solution in $(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$. By our assumption $f_{\mathcal{Y}}[S] > \eta(G, T, \mathcal{K})$. Then $\hat{f}_{\mathcal{Y}}[A] = \hat{h}_{\mathcal{Y}} = [A] = \beta(A)$ for each $A \in \mathcal{R}_{\mathcal{Y}}$, and the same number of compound S-paths as in $\hat{f}_{\mathcal{Y}}$ can be constructed by joining W-paths of $\hat{h}_{\mathcal{Y}}$ in $(G', T, \mathcal{R})$. Then $\eta(G', T, \mathcal{R}) > \eta(G, T, \mathcal{K})$ - a contradiction.

Therefore, if an inner node $x \notin \mathcal{X}$ exists, it lies on two S-paths of both $g_{\mathcal{X}}$ and $h_{\mathcal{X}}$. Let us $h_{\mathcal{X}}$-split x. For the resulting network $(G', T, \mathcal{K})$ there exists flat network $(G', T, \mathcal{R})$ with $\mathcal{R} \succeq \mathcal{K}$ and minimal dual solution $\mathcal{Y}$ for which Theorem 3.2 holds. Restoring x does not create an augmenting sequence in $\hat{h}_{\mathcal{Y}}$ because all W-paths of $\hat{h}_{\mathcal{X}}$ are edges as we have proved above. Then we have $\eta(G', T, \mathcal{R}) > \eta(G, T, \mathcal{K})$ - a contradiction. $\square$

Next, we need to show that the clutter $\mathcal{K}$ of a minimal counterexample network $(G, T, \mathcal{K})$ can be extended without increasing the maximum of the strong problem.

Proof of Theorem 3.2. Let $x \in X_t$, $t \in T$, be a pivot of a trident Q_1,Q_2, where Q_1 is an A_3-path and Q_2 is an (A_1, A_2)-path where A_1, A_2, A_3 are pairwise intersecting members of $\mathcal{K}$. Such a pivot always exists since $\hat{h}$ unlocks $\mathcal{K}$. Let $\mathcal{R} := \mathcal{K} \setminus A_i$ for $i \in \{1, 2, 3\}$ and let $\mathcal{Y}$ denote the minimal dual solution in $(G, T, \mathcal{R})$. We prove that $\eta(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}}) = \eta$. In $(G, T, \mathcal{K})$ we have $\eta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}}) > \eta$ by (4.5) and $N \setminus \mathcal{X} = \emptyset$ by Claim 4.4. We prove the claim by induction in the size of $N \setminus T$.

First, let us assume that $\mathcal{X}$ contains no inner nodes except x. Let $Q_1 := txP_3p_3$ and $Q_2 := p_1P_1xP_2p_2$, where $p_i \in A_i$ for $i = 1, 2, 3$. Then $\mathcal{Y} = T$ and we need to show that $\eta(G, T, \mathcal{R}) = \eta(G, T, \mathcal{K})$.

Recall that $g_{\mathcal{X}}$ denotes a common solution in $(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}})$. For $\eta(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}}) > \eta(G, T, \mathcal{K})$ to hold, $g_{\mathcal{X}}$ must contain paths $Y_tP_iR_i$ where R_i are B_i-paths of $\hat{h}$ with ends in p_i, $A_i \neq B_i \in \mathcal{K}$ for $i = 1, 2, 3$ and paths R_i are edges by Claim 4.4 (see Figure 6, Q_1 is dashed). Therefore, in any common solution in $(G, T, \mathcal{K})$ at most one path of R_1, R_2, R_3 is a subpath of an S-path.

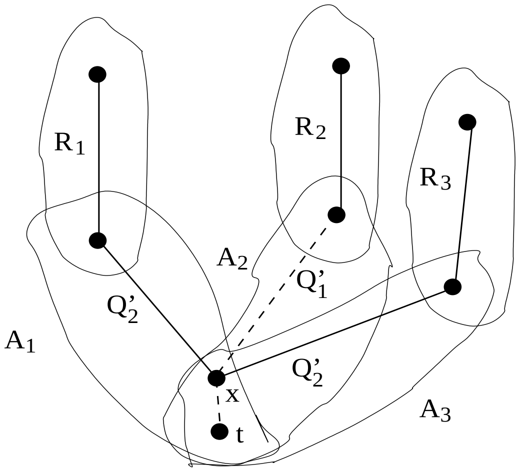


Figure 7: Paths Q_1' (dashed) and Q_2'.

Let us assume that $\eta(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}}) > \eta(G, T, \mathcal{K})$. Then h solves the weak problem in $(G, T, \mathcal{R})$ since it contains no augmenting sequences - the only paths of h with inner nodes are Q_1 and Q_2 which are S-paths in $(G, T, \mathcal{R})$. As restoring A_3 in $(G, T, \mathcal{R})$ decreases $f[S]$ by at most 1 (by turning Q_1 back into a W-path), we have $\eta(G, T, \mathcal{R}) = \eta(G, T, \mathcal{K}) + 1$. Then $f[S] = \eta(G, T, \mathcal{K})$ in $(G, T, \mathcal{K})$, and f is a common solution of $(G, T, \mathcal{K})$. Then at most one path of R_1, R_2, R_3, say R_1, is a subpath of an S-path of f in $(G, T, \mathcal{K})$. Since Q_1 is a W-path of f in $(G, T, \mathcal{K})$ by construction, we switch Q_1 and Q_2 in x so as to obtain an A_2-path Q_1' and an S-path Q_2' (see Figure 7). The resulting multiflow f' is also a common solution of $(G, T, \mathcal{K})$; however, joining Q_1' and R_2 creates a multiflow f'' with $f''[S] = f[S] + 1 = \eta(G, T, \mathcal{K}) + 1$ - a contradiction.

Same arguments apply when $\mathcal{R} := \mathcal{K} \setminus \{A_1\}$ or $\mathcal{R} := \mathcal{K} \setminus \{A_2\}$. Note that $(G, T, \mathcal{R})$ is integral by construction.

Let us now assume that $\mathcal{X}$ contains more than one inner node. Again, from Claim 4.4 it follows that $\hat{h}$ solves the weak problem in $(G, T, \mathcal{K})$. Since Q_1, Q_2 is not a trident in $(G, T, \mathcal{R})$ and $\mathcal{Y} \preceq \mathcal{X}$ by Corollary 4.3, we have $\mathcal{Y} \prec \mathcal{X}$. Let us select an inner node $y \in \mathcal{Y}$ such that $y \notin \mathcal{X}$. Since $\mathcal{X}$ is minimal dual solution, $\hat{h}_{\mathcal{Y}}$ unlocks $\mathcal{K}_{\mathcal{Y}}$ in $(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{K}_{\mathcal{Y}})$, thus the paths of $\hat{h}_{\mathcal{Y}}$ can be switched so as to obtain a trident with a pivot in some inner node x_n. We have $x_n \in \mathcal{X}$ and thus we select $y = x_n$. Therefore, any split of y preserves $\theta(G, T, \mathcal{K})$, and we can split y so that $\eta(G'_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}}) > \eta(G, T, \mathcal{K}) \geq \eta(G', T, \mathcal{K})$ for the resulting network $(G', T, \mathcal{K})$. By our assumption, $\eta(G'_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}}) = \eta(G', T, \mathcal{K})$. Let $g_{\mathcal{Y}}$ denote a common solution in $(G'_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$. Since $y \notin \mathcal{Y}$, gluing y back leaves $g_{\mathcal{Y}}$ a weak problem solution in $(G, T, \mathcal{R})$, and therefore $\hat{g}_{\mathcal{Y}}$ locks $\mathcal{Y}$ and $\mathcal{R}_{\mathcal{Y}}$ and $(G, T, \mathcal{R})$ is integral. Additionally, any common solution in $(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$ contains the same number of A-paths for all $A \in \mathcal{R}_{\mathcal{Y}}$, thus a common solution in $(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$ cannot have more compound S-paths than a common solution in $(G'_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}})$. Then $\eta(G_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}}) = \eta(G'_{\mathcal{Y}}, \mathcal{Y}, \mathcal{R}_{\mathcal{Y}}) \leq \eta$, as required. □

5. Good characterization

In this section, we prove that the dual combinatorial structure of Section 3 gives *good characterization for the fractional strong problem* in flat networks and *good characterization for the strong problem* in integral flat networks. It is worth noting that no adequate criterion for integrality in K-networks in known to date.

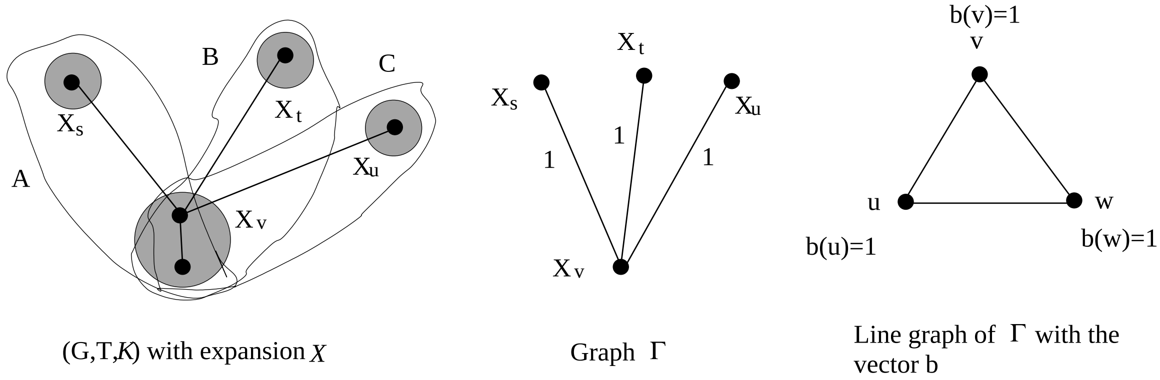


Figure 8: An example of graphs Γ and $L(\Gamma)$.

Let us observe a flat network $(G,T,\mathcal{K})$, flat clutter $\mathcal{R} \succeq \mathcal{K}$ and a dual solution $\mathcal{X}$ in $(G,T,\mathcal{R})$ where $\eta(G_\mathcal{X},\mathcal{X},\mathcal{R}_\mathcal{X}) = \eta(G,T,\mathcal{K})$. We define multigraph $\Gamma = (\mathcal{X}, E_\Gamma)$ by taking $\mathcal{X}$ as node set and adding edge (X_t,X_s) of multiplicity $\mathrm{mul}(X_t,X_s) = \beta_A$ for every pair $\{X_t,X_s\} \in \mathcal{R}$. We denote by $L(\Gamma) = (E_\Gamma, L)$ the line graph of Γ and define a vector $b = (\mathrm{mul}(e) | e \in E_\Gamma)$ on the vertices of $L(\Gamma)$. We denote by $m(\Gamma)$ the size of maximum b-matching in the graph $L(\Gamma)$ w.r.t vector b.

Figure 8 shows an example of a flat network with expansion, its corresponding graph Γ and the line graph $L(\Gamma)$ together with the vector b.

We can now define function φ on a network $(G,T,\mathcal{R})$, expansion $\mathcal{X}$ and graph $L(\Gamma)$:

$$\varphi(\mathcal{X},\mathcal{R}) := \frac{1}{2}\sum_{t\in T}\lambda(X_t) - \sum_{A\in\mathcal{R}}\beta(A) + m(\Gamma). \tag{5.6}$$

To show that the path packing problem in integral flat networks is in co-NP, we prove the following.

Theorem 5.1. *For multiflows f in a flat network $(G,T,\mathcal{K})$, flat clutters $\mathcal{R} \succeq \mathcal{K}$ and expansions $\mathcal{X}$ in $(G,T,\mathcal{R})$, equation*

$$\max_f f[S] = \min_{\mathcal{R}\succeq\mathcal{K},\mathcal{X}} \varphi(\mathcal{X},\mathcal{R})$$

gives good characterization for the fractional strong problem and good characterization for the strong problem if $(G,T,\mathcal{K})$ is integral.

Proof. Let us first show that $f[S] \le \varphi(\mathcal{X},\mathcal{R})$ for all extensions $\mathcal{R}$ of $\mathcal{K}$, expansions $\mathcal{X}$ in $(G,T,\mathcal{R})$ and multiflows f in a flat network $(G,T,\mathcal{K})$. Theorem 3.1 ensures that $\eta^{\mathrm{FR}}(G,T,\mathcal{K}) \le \eta^{\mathrm{FR}}(G_\mathcal{X},\mathcal{X},\mathcal{R}_\mathcal{X})$. We prove the claim for integral flat network $(G,T,\mathcal{K})$ (one can always multiply the edges of E in order to obtain an integral network from a non-integral one). Let $h_\mathcal{X}$ denote a common solution in $(G_\mathcal{X},\mathcal{X},\mathcal{R}_\mathcal{X})$ and let $m(h_\mathcal{X})$ denote the number of compound S-paths in $h_\mathcal{X}$. Then, since $\eta(G,T,\mathcal{K}) \le \eta(G,T,\mathcal{R})$, we have

$$\eta(G,T,\mathcal{K}) \le \frac{1}{2}\sum_{t\in T}\hat{h}_\mathcal{X}[X_t,X_t^c] - \sum_{A\in\mathcal{K}}\hat{h}_\mathcal{X}[A] + m(h_\mathcal{X}).$$

We need to show that:

$$\frac{1}{2}\sum_{t\in T}\lambda(X_t) - \sum_{A\in\mathcal{R}}\beta(A) + m(\Gamma) \ge \frac{1}{2}\sum_{t\in T}\hat{h}_\mathcal{X}[X_t,X_t^c] - \sum_{A\in\mathcal{R}}\hat{h}_\mathcal{X}[A] + m(h_\mathcal{X}).$$

By definition of λ, $\frac{1}{2}\sum_{t\in T}\hat{h}_{\mathcal{X}}[X_t, X_t^c] - \frac{1}{2}\sum_{t\in T}\lambda(X_t) \geq 0$. Let us now show by induction in the number of W-paths in $\hat{h}_{\mathcal{X}}$ that

$$m(\Gamma) - \sum_{A\in\mathcal{R}}\beta(A) \geq m(h_{\mathcal{X}}) - \sum_{A\in\mathcal{R}}\hat{h}_{\mathcal{X}}[A]. \tag{5.7}$$

If $\hat{h}_{\mathcal{X}}[A] = \beta(A)$ for all $A \in \mathcal{R}_{\mathcal{X}}$, we are done because $m(\Gamma) \geq m(h_{\mathcal{X}})$ by the maximality of $m(\Gamma)$. Otherwise, let us select $A \in \mathcal{R}_{\mathcal{X}}$ for which $\hat{h}_{\mathcal{X}}[A] > \beta(A)$. We remove the edges of an A-path P of h from the network so that $\beta(A)$ remains intact. Inequality (5.7) holds for the resulting network. The right side of the inequality does not change when we add the edges of P back, while $\sum_{A\in\mathcal{R}}\hat{h}_{\mathcal{X}}[A]$ grows by 1. Since $m(h_{\mathcal{X}})$ can increase by at most 1, the inequality (5.7) holds.

Theorem 3.2 proves the existence of a clutter $\mathcal{R} \succeq \mathcal{K}$ and an expansion $\mathcal{X}$ in $(G, T, \mathcal{R})$ so that $\eta(G, T, \mathcal{K}) = \max_f f[S] = \varphi(\mathcal{R}, \mathcal{X})$. Indeed, the number of simple S-paths in a common solution $h_{\mathcal{X}}$ in $(G_{\mathcal{X}}, \mathcal{X}, \mathcal{R}_{\mathcal{X}})$ is $\frac{1}{2}\sum_{t\in T} d(X_t) - \sum_{A\in\mathcal{R}}\beta(A)$, because $\mathcal{X}$ is a dual solution and $\hat{h}_{\mathcal{X}}$ locks $\mathcal{X}$ and $\mathcal{R}_{\mathcal{X}}$. The maximum number of compound S-paths in $h_{\mathcal{X}}$ is precisely $m(\Gamma)$, and since $h_{\mathcal{X}}[A] = \beta[A]$ for all $A \in \mathcal{R}_{\mathcal{X}}$, the equality holds.

We now have a good characterization for the fractional strong problem in flat networks, since function $\varphi(\mathcal{X}, \mathcal{R})$ is computable in polynomial time in the sizes of $\mathcal{X}$, $\mathcal{R}$ and G. Computing all $\lambda(X_t)$ and $\beta(A)$ requires computing sizes of $|T|+|\mathcal{R}|$ mincuts in $(G, T, \mathcal{R})$, which is a polynomial task by [FF 1956, Di 1970]. Computing a b-matching in a graph (which gives us $m(\Gamma)$), is a polynomial task as well (see, e.g., [AN 1987]). □

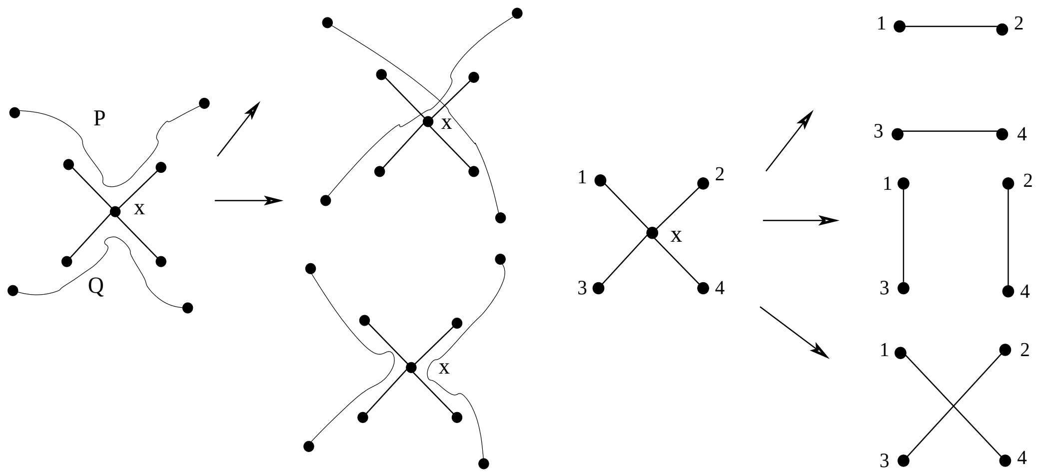
6. Conclusions

In this paper we prove the existence of a good characterization (otherwise called combinatorial max-min) for the path packing problem in a subclass of K-networks. Theorem 3.2 gives good characterization for the fractional path packing problem in flat networks and good characterization for the (integer) path packing problem in integral flat networks, showing that the path packing problem for integral flat networks is in co-NP. The class of flat networks covers almost all classes of network for which good characterization have been previously discovered, and has a non-trivial intersection with networks where each terminal is covered by at most two anticlique clutter members. One should note that all the above previously known network classes are integral. Additionally, flat networks is the first class of networks for which method of terminal expansions is not sufficient as dual combinatorial structure, and a new structure that uses clutter extensions needs to be used.

7. Appendix

7.1. Operations on paths and locking

Let paths P and Q of a multiflow f traverse an inner node x, so that $P = P'xP''$ and $Q = Q'xQ''$. *Switching* P and Q in x transforms them into paths $K = P'xQ'$ and $L = P''xQ''$, and f into the multiflow $f \setminus \{P, Q\} \cup \{K, L\}$. If $d(x) = 4$, there exists two different ways to switch P and Q (see Figure 9(a)). A *split* of an inner node x is a graph transformation consisting of removal of x and linking its neighbors by $\frac{d(x)}{2}$ edges so as to preserve their degrees. Given a multiflow h in a network, an *h-split* of an inner node is a split preserving the paths of h. There are three



(a) Switching paths in x.

(b) Splitting x.

Figure 9: Switching and splitting at inner node..

different ways in which a degree 4 node can be split, as Figure 9(b) shows. A multiflow obtained from a multiflow f by "breaking" its compound paths in inner terminals until no compound paths remain is denoted by $\hat{f}$.

7.2. Metric properties of switching

Metric properties of K-networks imply following useful properties of multiflows solving the weak problem w.r.t the switching operation (see, e.g., [V 2007] for detailed proofs). Let P and Q be two paths of a multiflow f that traverse an inner node x. P and Q form a *cross* if their ends differ.

Property 7.1. *If f solves the weak, P is an A-path, Q is an A^c-path, $A \in \mathcal{K}$, and P and Q form a cross, any switch of P and Q in x preserves $\Theta(f)$ (Clutter members B and C in Figure 10(a) ensure this).* □

Property 7.2. *If f solves the weak problem, P and Q are A-paths, $A \in \mathcal{K}$, and P and Q form a cross, then any switch of P and Q in x preserves $\Theta(f)$ (see Figure 10(b)).* □

Property 7.3. *If f solves the weak problem, P is an A-path, Q is a B-path, $A, B \in \mathcal{K}$ are distinct, and P and Q have exactly one common end, then any switch of P and Q in x preserves $\Theta(f)$ (see Figure 10(c)).* □

When a pair of paths P and Q satisfies conditions of Properties 7.1 or 7.3, it is called a *trident*; called simple trident in the latter case. A node x of a trident P, Q is called a *pivot*.

Property 7.4. *If f solves the weak problem, P is an A-path, Q is an (A, A^c)-path, $A \in \mathcal{K}$, then at least two switches of P and Q in x preserve $\Theta(f)$ (two clutter members B and C in Figure 10(d) cannot co-exist in a K-network).* □

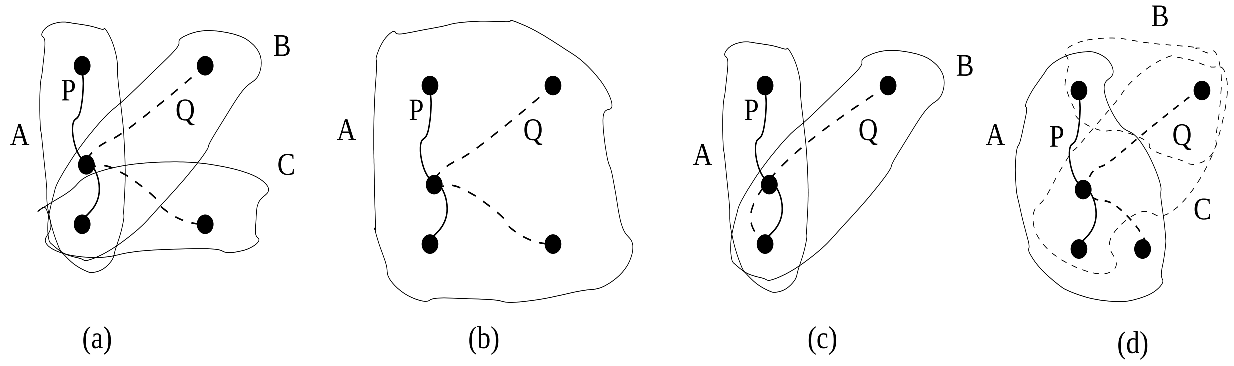


Figure 10: Switching in K-networks.

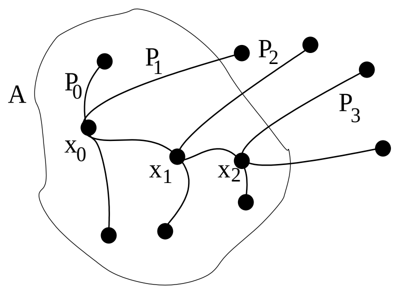


Figure 11: An example of augmenting sequence $P_0, x_0, P_1, x_1, P_2, x_2, P_3$.

7.3. Locking

A maximum multiflow f *locks* a subset $A \subseteq T$ if it contains a maximum (A, A^c)-flow, that is, if $f[A, A^c] = \lambda(A)$. Otherwise, f *unlocks* A. In other words, f locks A if it contains the smallest possible number of A-paths. A. Karzanov and M. Lomonosov have introduced in [KL 1978] the following application of the Ford-Fulkerson augmenting path procedure, assuming that a multiflow traverses each edge. A maximum multiflow unlocks $A \in \mathcal{K}$ if and only if it contains an *augmenting sequence* $P_0, x_0, ..., x_{i-1} P_i x_i, ..., P_n$ of paths P_0 (an A-path), $P_1, ..., P_{n-1}$ ((A, A^c)-paths), P_n (an A^c-path), and inner nodes $x_0, ..., x_{n-1}$ so that $x_i \in P_i, P_{i+1}$ for $i \in 0, ..., n-1$ and x_i is located on P_i between x_{i-1} and the A-end of P_i. Figure 11 shows an example of augmenting sequence.

In this paper, we use the fact that unlocking a member of $\mathcal{K}$ and existence of the alternating sequence are equivalent. When $\mathcal{K}$ is a K-clutter, there exists a series of switches of $P_0, ..., P_n$ in $x_0, ..., x_{n-1}$ creating a maximum multiflow f' with a cross and having $\Theta(f') \geq \Theta(f)$. If f solves the W-problem and unlocks $A \in \mathcal{K}$, switching $P_0, ..., P_{n-1}$ in $x_0, ..., x_{n-2}$ creates a multiflow f' with a trident P'_0, P'_1 with pivot x_{n-1}, where P'_0 is an A-path and P'_1 is an A^c-path.

7.4. Dual solutions of the weak problem

7.4.1. A max-min theorem for the weak problem

Let $\mathcal{X} = \{X_t \subseteq N \mid t \in T\}$ be an expansion of T in a network $(G, T, \mathcal{K})$, where each member of $\mathcal{X}$ contains a single terminal. Set $\mathcal{X}$ taken as terminals together with clutter $\mathcal{K}_{\mathcal{X}} = \{A_{\mathcal{X}} = \{X_t \in \mathcal{X}\}_{t \in A} | A \in \mathcal{K}\}$ on $\mathcal{X}$ for a network $(G_{\mathcal{X}}, \mathcal{X}, \mathcal{K}_{\mathcal{X}})$. Let us call a pair of terminals in a clutter *weak* or *strong* if that pair belongs to class W or S w.r.t. to that clutter. An $\mathcal{X}$-*path* in G is an (x, y)-path with x, y lying in distinct members of $\mathcal{X}$. An $\mathcal{X}$-*flow* is a

multiflow in the network $(G_\mathcal{X}, \mathcal{X}, \mathcal{K}_\mathcal{X})$ consisting of $\mathcal{X}$-paths. Maxima of strong problem and weak problem in $(G_\mathcal{X}, \mathcal{X}, \mathcal{K}_\mathcal{X})$ are denoted by $\eta_\mathcal{X}$ and $\theta_\mathcal{X}$ respectively.

Let $\mathcal{X} = (X_t : t \in T)$ and $\mathcal{Y} = (Y_t : t \in T)$ be expansions. Note that for every $\mathcal{X} \preceq \mathcal{Y}$, every $\mathcal{X}$-flow is also a $\mathcal{Y}$-flow (but the converse may be not true). Since for $\mathcal{X} \preceq \mathcal{Y}$ any $\mathcal{X}$-flow is also a $\mathcal{Y}$-flow, $\theta_\mathcal{Y} \geq \theta_\mathcal{X}$. Since T-flow is also an $\mathcal{X}$-flow, $\theta_\mathcal{X} \geq \theta$. We call an expansion $\mathcal{X}$ *critical* if $\theta_\mathcal{Y} > \theta_\mathcal{X}$ for every $\mathcal{Y} \succ \mathcal{X}$. A critical $\mathcal{X}$ with $\theta_\mathcal{X} = \theta$ is called a *dual solution*. The triangle theorem ([L 1985]) ensures that

$$\textit{there exists a maximum } \mathcal{X}\textit{-flow } h \textit{ such that } \Theta_\mathcal{X}(h) = \theta_\mathcal{X}. \tag{7.8}$$

We limit ourselves to K-networks $(G, T, \mathcal{K})$ with simple $\mathcal{K}$. The results of this section that hold for simple clutters hold for general K-networks as well, because compressing equivalent terminals does not change θ by the triangle theorem from [L 1985]. Then (7.8) implies that for a maximum X-flow h (even when $\mathcal{X} = T$)

$$\Theta_\mathcal{X}(h) = |h| - \frac{1}{2} h[W]. \tag{7.9}$$

We aim to prove the following max-min theorem for the fractional W-problem.

Theorem 7.5. *In a K-network $(G, T, \mathcal{K})$*

$$\max{}_f \Theta(f) = \min{}_\mathcal{X}\left(\tfrac{1}{2} \textstyle\sum_{t \in T} d(X_t) - \tfrac{1}{2} \sum_{A \in \mathcal{K}_\mathcal{X}} \beta(A)\right). \tag{7.10}$$

The maximum is taken over the fractional multiflows in $(G, T, \mathcal{K})$, and the minimum is taken over all expansions in $(G, T, \mathcal{K})$. Moreover, (7.10) *holds as equality for every dual solution $\mathcal{X}$.*

To prove this theorem, we state the following inequality for an expansion $\mathcal{X}$ and a T-flow f:

$$\Theta(f) \overset{(a)}{\leq} \theta \overset{(b)}{\leq} \Theta_\mathcal{X}(h) \overset{(c)}{\leq} \frac{1}{2} \sum_{t \in T} d(X_t) - \frac{1}{2} \sum_{A \in \mathcal{K}_\mathcal{X}} \beta(A) \tag{7.11}$$

We aim to show that (7.11) holds as inequality for every expansion and as equality for every critical expansion. (7.11)(a) follows directly from the definition of θ. (7.11)(b) holds because f is also an $\mathcal{X}$-flow. (7.11)(c) holds because there exists a maximum $\mathcal{X}$-flow h that solves the weak problem in $\mathcal{X}$. For such h the minimum of $\sum_{A \in \mathcal{K}_\mathcal{X}} h[A]$ is achieved when all $A \in \mathcal{K}_\mathcal{X}$ are locked by h, i.e. $\sum_{A \in \mathcal{K}_\mathcal{X}} h[A] \leq \sum_{A \in \mathcal{K}_\mathcal{X}} \beta(A)$ and $|h| = \frac{1}{2} \sum_{t \in T} \lambda(X_t)$ by the Lovăsz-Cherkassky theorem ([Lo 1976, Ch 1977]). We need the following two claims to show that (7.11)(c) is an equality.

Claim 7.6. *Let $\mathcal{X}$ be a dual solution in a simple K-network $(G, T, \mathcal{K})$. A maximum fractional $\mathcal{X}$-flow h that satisfies $\Theta_\mathcal{X}(h) = \theta_\mathcal{X}$ (that is, solves the weak problem in $(G_\mathcal{X}, \mathcal{X}, \mathcal{K}_\mathcal{X})$) locks X_t for all $t \in T$.*

Proof. First, let us show that h saturates every $(X_t, \overline{X_t})$-edge. Let e be an (x, y)-edge with $x \in X_t$ and $y \in \overline{X_t}$. Let $\mathcal{Y} \succ \mathcal{X}$ be an expansion where $Y_s = X_s$ for terminal $s \neq t$ and $Y_t = X_t \cup \{y\}$. Since $\mathcal{X}$ is critical, $\theta_\mathcal{Y} > \theta_\mathcal{X}$ and there exists a $\mathcal{Y}$-flow g such that $\Theta_\mathcal{Y}(g) > \theta_\mathcal{X}$. Let us denote the unused capacity of e by ε and let $\delta = g[y, \cup_{s \neq t} X_s]$. Clearly, $\varepsilon < \delta$. We turn g into an $\mathcal{X}$-flow by prolonging all its paths starting in y to x instead through the edge e. Let g'

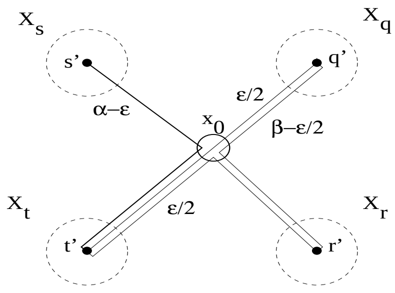


Figure 12: $\frac{3}{2}$-operation.

be the functions on $\mathcal{X}$-paths thus obtained; g' does not satisfy the capacity constraint on (x, y). Then there exists $0 < \alpha < 1$ such that $h' = (1-\alpha)h + \alpha g'$ is an $\mathcal{X}$-flow. h' satisfies all capacity constraints and has $\Theta_{\mathcal{X}}(h') \geq (1-\alpha)\Theta_{\mathcal{X}}(h) + \alpha\Theta_{\mathcal{Y}}(g) > \theta_{\mathcal{X}}$, contradicting the definition of $\mathcal{X}$.

Let us assume now that a (p, q)-path P of h, $p \in X_t$, contains two $(X_t, \overline{X_t})$-edges, $e_1 = (x_1, y_1)$ and $e = (x_2, y_2)$ where $x_1, x_2 \in X_t, y_1, y_2 \in \overline{X_t}$ and y_1, x_1, x_2, y_2 appear on P in this order. Then by replacing P with x_2Pq we obtain an $\mathcal{X}$-flow g for which $\Theta_{\mathcal{X}}(g) = \theta_{\mathcal{X}}$ and the edge (x_1, y_1) is not saturated by g, a contradiction. □

Claim 7.7. *Let $\mathcal{X}$ be a dual solution in a simple K-network $(G, T, \mathcal{K})$. A maximum fractional $\mathcal{X}$-flow h would then satisfy $\Theta_{\mathcal{X}}(h) = \theta_{\mathcal{X}}$ iff every $A \in \mathcal{K}_{\mathcal{X}}$ is locked by h.*

Proof. The "if" direction is trivial. Let h be a maximum $\mathcal{X}$-flow with $\Theta_{\mathcal{X}}(h) = \theta_{\mathcal{X}}$ that locks every member of $\mathcal{K}_{\mathcal{X}}$. Because of Claim 7.6 and the simplicity of $\mathcal{K}_{\mathcal{X}}$, we get
$\Theta(h) = \frac{1}{2}\sum_{X\in\mathcal{X}} d(X) - \frac{1}{2}\sum_{A\in\mathcal{K}_{\mathcal{X}}} \beta_A$ and thus $\Theta(h) \geq \theta_{\mathcal{X}}$ by (7.11)(c).

For the "only if" direction, assume that h is a maximum $\mathcal{X}$-flow that has $\Theta_{\mathcal{X}}(h) = \theta_{\mathcal{X}}$ and unlocks $A \in \mathcal{K}_{\mathcal{X}}$. Let A^c in the context of $\mathcal{K}_{\mathcal{X}}$ denote the members of $\mathcal{X}$ that do not lie in A. Then h contains an augmenting sequence $P_0, x_0, ..., x_{m-1}, P_m$, where P_0 is an A-path, P_m is an A^c-path, and each one of $P_1, ..., P_{m-1}$ is an (A, A^c)-path. We can choose h so that $m = 1$. Let P_0 and P_1 be (s', t')- and (q', r')-paths with weights α and β respectively where $s' \in X_s$, $t' \in X_t$, $q' \in X_q$ and $r' \in X_r$. Since a switch of P_0 and P_1 in x_0 cannot increase $\Theta(h)$, we can assume that w.l.o.g. (X_q, X_r), (X_t, X_r) and (X_t, X_q) are S-pairs while (X_s, X_q) and (X_s, X_r) are W-pairs by the simplicity of $\mathcal{K}_{\mathcal{X}}$.

We construct a new flow f from h by replacing P_0 and P_1 with (t', r'), (t', q'), (q', r') and (s', t')-paths of weights $\frac{\varepsilon}{2}$, $\frac{\varepsilon}{2}$, $\beta - \frac{\varepsilon}{2}$ and $\alpha - \varepsilon$ respectively (we call this operation $\frac{3}{2}$-*operation*, see Figure 12). It follows that $|f| = |h| - \frac{\varepsilon}{2}$ and $f[W] = h[W] - \varepsilon$ since $(X_q, X_t), (X_q, X_r), (X_r, X_t) \in S$ and $\Theta_{\mathcal{X}}(f) = \Theta_{\mathcal{X}}(h)$.

The subpath $s'P_0x_0$ does not have common nodes with any other $\mathcal{X}$-path Q whose ends do not lie in $X_s \cup X_t$. If it were so, then the above $\frac{3}{2}$-operation could be applied to both P_0, P_1 and P_0, Q and a flow f' with $|f'| = |h| - \frac{\varepsilon}{2}$ and $f'[W] = h[W] - 2\varepsilon$ could be created, which contradicts the maximality of $\Theta_{\mathcal{X}}(h)$. Therefore, there exists an edge (s', x) of $s'Lv$ which is not saturated by f - a contradiction to Claim 7.6. □

Theorem 7.5 follows from Claims 7.6 and 7.7.

7.5. Properties of dual solutions

A *mixed flow* among flows $h_1, ..., h_n$ is a flow $h := a_1h_1 + ... + a_nh_n$ where $a_1, ..., a_n$ are positive rational numbers and $a_1 + ... + a_n = 1$. A mixed flow among *all fractional maximum flows*

solving the weak problem is called the *mixed solution.*

Claim 7.8. *A K-network* $(G, T, \mathcal{K})$ *has unique minimal dual solution.*

Proof. Let h be a mixed solution and let E_0 be a subset of edges not saturated by h and reachable from T. Let $\mathcal{X}$ be the expansion of T consisting of all $t \in T$ and E_0. Clearly, all the $(X_t, \overline{X_t})$-edges, $t \in T$, are saturated by h.

Let us assume that h unlocks $A \in \mathcal{K}_{\mathcal{X}}$. Then h contains an augmenting sequence for A and there exists a series of switches that produces a maximum flow h' maximizing Θ. h' has an (X_s, X_t)-path and an (X_p, X_q)-path with a common node, where $X_s, X_t \in A$ and $X_p, X_q \in A^c$, $(X_p, X_q) \in S$. Applying a $\frac{3}{2}$-operation to these paths, we obtain a maximum flow h'' that maximizes Θ and does not saturate one of the $(X_t, \overline{X_t})$-edges - a contradiction. Therefore, $\mathcal{X}$ is a dual solution.

Let $\mathcal{Y}$ be a dual solution that does not include $e = (u, v) \in E_0$ and f be a maximum $\mathcal{Y}$-flow. Since e is not saturated by h, a combined flow $f = \alpha g + (1 - \alpha)h$, $0 < \alpha < 1$ is a maximum $\mathcal{Y}$-flow that does not saturate e either. It follows from the definition of E_0 that there exists $t \in T$ and a path P from T to e, whose edges lie in E_0. By the choice of E_0, the edges of P are not saturated by f, and one of those edges is an $(Y_t, \overline{Y_t})$-edge - a contradiction. $\square$

Dual solutions have the following property with respect to the pivots of tridents.

Claim 7.9. *Let* $\mathcal{X}$ *be a dual solution and* h *be a maximum* $\mathcal{X}$*-flow in a K-network* $(G, T, \mathcal{K})$*. Then* $\mathcal{X}$ *contains the pivots of all the tridents in* h*.*

Proof. Let (P, Q, x) be a trident of h and assume that $\mathcal{X}$ does not contain x. Let P be a (t, s)-path, $t, s \in T$. Suppose first that (P, Q, x) is an ordinary trident. Then a $\frac{3}{2}$-transformation of P, Q in x creates a fractional multiflow h' with $\Theta_{\mathcal{X}}(h') = \Theta_{\mathcal{X}}(h)$ and unsaturated edges in tPx, which contradicts Claim 7.6.

Assume now that (P, Q, x) is a simple trident, which means that P and Q are (s, u_1)- and (t, u_2)-paths respectively, with $s \sim t$ and $s, u_1 \in A \in \mathcal{K}_{\mathcal{X}}$ while $t, u_2 \in B \in \mathcal{K}_{\mathcal{X}}$, $A \neq B$. We can obtain a new flow h' from h by replacing P and Q with $u_1 PxQu_2$. $\Theta_{\mathcal{X}}(h') = \Theta_{\mathcal{X}}(h)$ and the edges of tPx and tQx are not be saturated by h' - a contradiction. $\square$

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